ISOMETRIES OF DIAGONALLY SYMMETRIC BANACH SPACES

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ABSTRACT
We describe all compact symmetric subgroups of the orthogonal group O_n which contain the permutation group S_n. There are seven such groups, each one can be realised as the group of all isometries on some n-dimensional Banach space.

Introduction

Let $(E, \| \|)$ be a normed finite-dimensional space over the reals. If $B = \{b_i, b_i'\}_{i \leq n}$ is a basis for E, the diagonal asymmetry constant of B is defined as

$$\delta(B) = \sup\left\{ \left\| \sum_{i=1}^{n} x_i b_{\Pi(i)} \right\| \Big/ \left\| \sum_{i=1}^{n} x_i b_i \right\| \right\}$$

where the supremum is taken over all vectors and permutations Π of the set $\{1, 2, \cdots, n\}$. The unconditional asymmetry constant of B is defined as

$$\chi(B) = \sup\left\{ \left\| \sum_{i=1}^{n} \varepsilon_i x_i b_i \right\| \Big/ \left\| \sum_{i=1}^{n} x_i b_i \right\| \right\}$$

where the supremum is taken over all vectors and signs $\varepsilon_i = \pm 1$, $1 \leqq i \leqq n$. The total asymmetry constant of B is defined by

$$t(B) = \sup\left\{ \left\| \sum_{i=1}^{n} \varepsilon_i x_i b_{\Pi(i)} \right\| \Big/ \left\| \sum_{i=1}^{n} x_i b_i \right\| \right\}$$

where the supremum is taken over all vectors, signs $\varepsilon_i = \pm 1$ and permutations Π of $\{1, 2, \cdots, n\}$.

If p denotes one of the symbols δ, χ or t, $p(E)$ is defined to be $\inf p(B)$, the infimum is taken over all bases B for E. E is called diagonally (respectively,

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unconditionally, totally) symmetric if $\delta(E)$ (respectively, $\chi(E)$, $t(E)$) is equal to 1.

The reader may refer to [1, 2, 3, 4, 5, 6] for many results on these and related asymmetry constants. Let us briefly mention that it was recently proved in [6] that $t(E) \leqq 9\delta(E)$, and it is also known that none of the asymmetry constants defined above is uniformly bounded.

In this paper we investigate diagonally symmetric spaces E, namely, spaces with $\delta(E)=1$. It turns out that if E is diagonally symmetric with dimension $n \geqq 13$, then the group of isometries of E, G, has a structure which can be given a complete description. One consequence of this description is that if G is a finite group then it must consist of exactly $2^n[n!]$, $2^{n-1}[n!]$, $2[(n+1)!]$, $4[n!]$ or $2[n!]$ elements. If G is infinite, either $G = O_n$ (= the group of orthogonal $n \times n$ matrices) or $G = O_{n-1} \oplus O_1$.

The algebraic version is the description of all compact subgroups of O_n, which are symmetric (i.e. contain $-I$) and contain the permutation group S_n. We also obtain the characterization of all diagonally symmetric bases for a given finite-dimensional normed space.

§1. G is infinite

Let $(E, \| \|)$ be a real n-dimensional normed space, and $(,)$ be the inner product generated by the ellipsoid of least volume enclosing the unit sphere $M = \{x \in E;\ \|x\| = 1\}$. Set $\|x\|_2 = \sqrt{(x,x)}$ for $x \in E$.

If $F \subseteq E$ is any subspace, $F^\perp$ will denote the orthocomplement of F with respect to the $\| \|_2$ norm. Let G denote the group of isometries of E. Since the minimal ellipsoid is unique, each $g \in G$ is also an orthogonal transformation.

Lemma 1.1. *If $\delta(E)=1$ then E has an orthonormal basis B with $\delta(B)=1$.*

Proof. Let $\{e_i\}_{i \leqq n}$ be any basis with $\delta(\{e_i\}) = 1$ and write $e = \sum_{i=1}^n e_i$. Every permutation of indices is a $(,)$ orthogonal transformation so $(e, e_1) = n^{-1}\|e\|_2^2$ and

$$\|e\|_2^2 = n\|e_1\|_2^2 + n(n-1)(e_1, e_2).$$

Thus for $i \neq k$ the equation

$$0 = (e_i + \mu e, e_k + \mu e) = \mu^2 \|e\|_2^2 + 2\mu(e, e_1) + (e_1, e_2)$$

has a solution if

$$0 \leqq 4(e, e_1)^2 - 4(e_1, e_2)\|e\|_2^2 = 4n^{-1}\|e\|_2^2[\|e_1\|_2^2 - (e_1, e_2)].$$

But clearly $(e_1, e_2) \leqq \| e_1 \|_2 \| e_2 \|_2 = \| e_1 \|_2^2$. The desired basis is $b_i = \lambda(e_i + \mu e)$, $1 \leqq i \leqq n$, where μ is chosen as above to make the b_i's orthogonal and λ so that $\| b_i \|_2 = 1$. □

Denote by $L(E)$ the space of all operators on E. If $a, b \in E$, $a \otimes b \in L(E)$ is defined by $a \otimes b(x) = (a, x)b$.

In this section we shall assume G is an infinite group, then it is an infinite compact Lie group, so there is a continuous homomorphism $g : (-\infty, \infty) \to G$, necessarily of the form $g(t) = \exp(tu)$, where $0 \neq u \in L(E)$ is fixed, $-\infty < t < \infty$, hence $g(t)$ is differentiable and $g^{-1}(t) = g^*(t) = g(-t)$, $g'(0) = u$.

LEMMA 1.2. *Assume G is infinite and $x \in E$ is such that $g'(0)x \neq 0$ and so that the set $G_x = \{g \in G; g(x) = x\}$ has as commutators only operators of the form $\lambda 1_E + \mu x \otimes x$ where λ and μ are arbitrary scalars. Then $\| \; \| = \| x \| \, \| x \|_2^{-1} \| \; \|_2$ on E.*

PROOF. Since $\| x \|_2^2 = (g(t)x, g(t)x)$

$$0 = \frac{d}{dt}(g(t)x, g(t)x) = 2(g'(t)x, g(t)x), \quad \text{so} \quad g'(0)x \in [x]^{\perp}.$$

Let us assume $\| x \| = 1$.

The assumption on G_x clearly implies each element of G_x maps $[x]^{\perp}$ to $[x]^{\perp}$. We claim also that if $0 \neq y \in [x]^{\perp}$ then $\operatorname{span}\{g(y); g \in G_x\} = [x]^{\perp}$. Assume not, then there exists $0 \neq z$ orthogonal to $g(y)$ for all $g \in G_x$. Let dg be the normalized Haar measure of the group G_x and $h = \int_{G_x} g(z) \otimes g(z)\, dg$. Then h commutes with each $g \in G_x$, so has the form $\lambda 1_E + \mu x \otimes x$. But $h(y) = h(x) = 0$, hence $h = 0$, contradicting trace $(h) = (z, z) \neq 0$.

There are therefore $(n-1)$ isometries $h_1, \cdots, h_{n-1} \in G$, where $\dim(E) = n$, with $h_i(x) = x$ for each i and $[h_i g'(0)x]_{i \leqq n-1} = [x]^{\perp}$. Define the map $\Phi : R^{n-1} \to E$ by

$$\Phi(t_1, \cdots, t_{n-1}) = (h_1 g(t_1) h_1^{-1}) \cdots (h_{n-1} g(t_{n-1}) h_{n-1}^{-1})x.$$

Then $\dfrac{\partial \Phi}{\partial t_k}(0, \cdots, 0) = h_k g'(0)x$, and so form a set of $n-1$ independent vectors. By the open mapping theorem, Φ maps homeomorphically an open neighbourhood of the origin in R^{n-1} onto an open subset of the unit sphere M, where "open" here means in the relative topology of M, about the point x.

Let $N = \{y \in M;\ y = g(x) \text{ for some } g \in G\}$. Then $N \subseteq M$ is closed and contains an open set about x, and so by translation about any $y \in M$, so N is also open, hence $N = M$. Given any $y \in M$, $y = g(x)$ for some $g \in G$, so $\| y \|_2 = \| x \|_2$, which concludes the proof. □

THEOREM 1.3. *Let $(E, \|\ \|)$ be a real n-dimensional normed space with $\delta(E) = 1$ and assume G is infinite. There are only two distinct possibilities:*

(1) *E is isometric to l_2^n.*

(2) *There is a vector b such that $[b]^\perp$ is isometric to l_2^{n-1}. Any orthogonal transformation of $[b]^\perp$, h, extends to an isometry $\tilde{h} \in G$ by setting $\tilde{h}(tb + y) = tb + h(y)$ $(t \in R^1, y \in [b]^\perp)$. Moreover, every $g \in G$ satisfies $g(b) = \pm b$.*

PROOF. By Lemma 1.1 there exists an orthonormal basis $\{b_i\}_{i=1}^n = B$ for E with $\delta(B) = 1$. There are two possibilities for $b = \Sigma_{i=1}^n b_i$,

$$(1) \quad g'(0)\,b \neq 0, \quad \text{or,} \quad (2) \quad g'(0)\,b = 0.$$

In case (1), the operators $g_{ij} = 1_E - (b_i - b_j) \otimes (b_i - b_j)$ for any i, j are in G_b, and only operators of the form $\lambda 1_E + \mu b \otimes b$ (λ, μ scalars) commute with all g_{ij}, so by Lemma 1.2 E is isometric to l_2^n.

In case (2), $g(t)b = b$ for all $-\infty < t < \infty$. Since $g'(0)b = 0$ and $g'(0) \neq 0$, and since the linear span of the set $\{nb_i - b;\ 1 \leqq i \leqq n\}$ is $[b]^\perp$, we may assume $g'(0)(nb_1 - b) \neq 0$. Let $x = (nb_1 - b)/\| nb_1 - b \|$, then $x \in [b]^\perp$ and for $i, j \geqq 2$ $g_{ij}(x) = x$, and g_{ij} maps $[x, b]^\perp$ onto $[x, b]^\perp$. As above $(g'(0)x, x) = 0$ and

$$(g'(0)x, b) = \frac{d}{dt}(g(t)x, b)\Big|_{t=0} = \frac{d}{dt}(x, g(-t)b)\Big|_{t=0} = -(x, g'(0)b) = 0$$

so $g'(0)x \in [x, b]^\perp$ and the set $\{g_{ij}g'(0)x;\ i \neq j \geqq 2\}$ spans $[x, b]^\perp$. Therefore there exist $(n-2)$ elements $h_i \in G$, $i = 1, 2, \cdots, n-2$, such that $h_i(x) = x$, $h_i(b) = b$, and $\{h_i g'(0)x\}_1^{n-2}$ spans $[x, b]^\perp$. Define $\Phi : R^{n-2} \to E$ by

$$\Phi(t_1, \cdots, t_{n-2}) = \prod_{i=1}^{n-2} h_i g(t_i) h_i^{-1} x.$$

As in Lemma 1.2, Φ maps homeomorphically an open subset of the origin in R^{n-2} onto an open subset (in the relative topology) of $M \cap [b]^\perp$. The proof is concluded as there to show that on $M \cap [b]^\perp$ $\|\ \|_2 = \| x \|_2$.

Let now h be any orthogonal transformation of $[b]^\perp$. Define $\tilde{h} \in L(E)$ by

$$\tilde{h}(tb + y) = tb + h(y) \qquad (t \in R^1, \quad y \in b^\perp).$$

The map $\Pi_{i=1}^{n-2} h_i g(t_i) h_i^{-1}$ maps $[b]^\perp$ onto $[b]^\perp$, is in G, hence orthogonal, so for any $y \in [b]^\perp$ there exists $g_y \in G$ such that $g_y(y) = h(y)$ and $g_y(b) = b$. Then

$$\| \tilde{h}(tb + y) \| = \| tb + h(y) \| = \| tb + g_y(y) \| = \| g_y(tb + y) \| = \| tb + y \|$$

which implies $\tilde{h} \in G$.

To prove the final statement assume to the contrary that

$g_0(b) = t_0 b + s_0 x_0$ for some $g_0 \in G$ where $x_0 \in [b]^\perp$, $\| x_0 \|_2 = 1$ and $s_0 \neq 0$.

Since $g'(0)b = 0$, there exists $y_0 \in [b]^\perp$, $\| y_0 \|_2 = 1$ and $g'(0)y_0 \neq 0$. Let h be an orthogonal transformation of $[b]^\perp$ such that $h(x_0) = y_0$ and let $x = \tilde{h}g_0(b) = t_0 b + s_0 y_0$. Then $g'(0)x \neq 0$.

Set $h_{ij} = (\tilde{h}g_0)g_{ij}(\tilde{h}g_0)^{-1}$. Then $h_{ij}(x) = x$ for every $i, j = 1, \cdots, n$, and only $\lambda 1_E + \mu x \otimes x$ are operators which commute with all h_{ij}, so by Lemma 1.2, E is isometric to l_2^n. Thus if E is not isometric to l_2^n, then $g(b) = \pm b$ for every $g \in G$. □

The asymmetry constant $s(E)$ ([1]) is the least λ such that there exists a group of isomorphisms G of E, $\| g \| \leqq \lambda$ for every $g \in G$, and the only operators which commute with all elements of G are the scalar multiples of the identity operator on E. If $s(E) = 1$, E is said to have enough isometries.

The distance coefficient between isomorphic Banach spaces E and F is defined as $d(E, F) = \inf \| T \| \, \| T^{-1} \|$, ranging over all $1-1$ operators T mapping E onto F.

Corollary 1.4. *If E is finite-dimensional, $\delta(E) = s(E) = 1$, and G is infinite, then E is isometric to a Hilbert space.*

Proof. Let $B = \{b_i\}_{i \leq n}$ be an orthonormal basis for E with $\delta(B) = 1$, and let $b = \sum_{i=1}^n b_i$. Let $g : (-\infty, \infty) \to G$ be some non-constant differentiable map. Since $s(E) = 1$, there exists $h \in G$ such that $g'(0)h(b) \neq 0$. Observe that $g'(0)h(b) \in [h(b)]^\perp$ and that $G_{h(b)}$ contains $h(\sum b_i \otimes b_{\Pi(i)})h^{-1}$ for any permutation Π of $\{1, 2, \cdots, n\}$. Therefore, by Lemma 1.2, taking $x = h(b)$, E is isometric to l_2^n. □

Corollary 1.5. *Let E be a real n-dimensional normed space. Then $d(E, l_2^n)$ is equal to the least scalar λ such that there exists an infinite group G of isomorphisms of E, $\| g \| \leqq \lambda$ for every $g \in G$, G contains the group of all permutations with respect to some basis of E and only the scalar multiples of 1_E are operators which commute with each $g \in G$.*

Proof. Obvious by Corollary 1.4. □

Corollary (Rolewicz [7]) 1.6. *If $t(E) = 1$ and G is infinite then E is isometric to a Hilbert space.*

Proof. Since $1 \leqq s(E)$, $\delta(E) \leqq t(E) = 1$. □

Denote by O_n the group of all orthogonal $n \times n$ matrices over the reals. We have the following algebraic version of Theorem 1.3.

COROLLARY 1.7. *Let G be any compact symmetric infinite group of $n \times n$ matrices over the reals which contains the group of all permutations of R^n. Then $G = O_n$ or $G = O_{n-1} \otimes O_1$.*

PROOF. Let $0 \neq e \in R^n$, and M be the convex hull of $\{\pm g(e); g \in G\}$. M is a centrally symmetric, closed convex body of R^n, so defines a norm on the space by

$$\| x \| = \inf\{t \geqq 0; x \in tM\}.$$

The space $E = (R^n, \| \ \|)$ satisfies $\delta(E) = 1$, and each $g \in G$ is an isometry of E, the result follows by Theorem 1.3. □

A real function $f(t)$ on the interval $[-1,1]$ is called convex if for every $-1 \leqq t_1 \leqq t \leqq t_2 \leqq 1$

$$[f(t) - f(t_1)](t_2 - t_1) \geqq [f(t_2) - f(t_1)](t - t_1).$$

f is called even if $f(t) = f(-t)$ for every t.

COROLLARY 1.8. *Let $(E, \| \ \|)$ be a real n-dimensional normed space and G be infinite. Then $\delta(E) = 1$ iff there exists a non-negative continuous, even, convex real function $f(t)$ defined on $[-1,1]$ with $f(0) > 0$, such that the unit ball of E is isometric to*

$$\left\{(x_1, \cdots, x_n) \in R^n; \left(\sum_1^{n-1} x_i^2\right)^{1/2} \leqq f(x_n), |x_n| \leqq 1\right\}.$$

PROOF. Obvious by Theorem 1.3. □

COROLLARY 1.9. *If $\dim(E) = n$, $\delta(E) = 1$ and G is infinite, then $d(E, l_2^n) \leqq \sqrt{2}$.*

PROOF. By Corollary 1.8 there is a function $f(t)$ such that the unit ball of E may be realised as

$$M = \left\{(x_1, x_2, \cdots, x_n); \left(\sum_1^{n-1} x_i^2\right)^{1/2} \leqq f(x_n), |x_n| \leqq 1\right\}.$$

Consider the 2-dimensional cross-section of M given by

$$M_2 = \{(x_1, 0, \cdots, 0, x_n); |x_1| \leqq f(x_n), |x_n| \leqq 1\}.$$

Let ε_2 be the ellipse of least area with center at the origin containing M_2. It is easy to see that $2^{-1/2}\varepsilon_2 \subseteq M_2 \subseteq \varepsilon_2$, and that the principal axes of ε_2 are orthogonal. Rotating M_2 and ε_2 through an $(n-1)$-dimensional space orthogonal to the x_n axis we obtain the unit ball M and an ellipsoid ε, with $2^{-1/2}\varepsilon \subseteq M \subseteq \varepsilon$, implying $d(E, l_2^n) \leqq \sqrt{2}$. □

COROLLARY 1.10. *Let $(E, \| \|)$ be a real infinite dimensional Banach space not isometric to l_2 and $B = \{e_i\}_{i \geqq 1}$ a basis for E satisfying $\delta(B) = 1$. Then there is an integer N such that if I is any finite subset of integers with $|I| > N$, the subspace $E_I = \operatorname{span}\{e_i\,;\, i \in I\}$ has a finite group of isometries.*

PROOF. Assume the assertion to be false. Then for any integer $k \geqq 1$ there is a finite subset $I_k = \{p_{k,1}, p_{k,2}, \cdots, p_{k,n_k}\}$ of n_k distinct integers, where $n_k \to \infty$, such that the group of isometries of E_{I_k} is infinite. Define $U_k : E \to E_{I_k}$ by

$$U_k\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{n_k - 1} x_i e_{p_{k,i}} - (n_k - 1)^{-1}\left(\sum_{i=1}^{n_k - 1} x_i\right) e_{p_{k,n_k}}.$$

Theorem 1.3 and proof imply that $U_k(E)$ is isometric to $l_2^{n_k - 1}$; further, if $\| \|_k$ denotes the Hilbert norm generated by the ellipsoid of minimal volume enclosing the unit ball of E_{I_k}, then on $U_k(E)$ the norms satisfy the equation

$$\begin{aligned} \| \cdot \| &= \| e_{p_{k,1}} - e_{p_{k,2}} \| \, \| e_{p_{k,1}} - e_{p_{k,2}} \|_k^{-1} \| \cdot \|_k \\ &= \| e_1 - e_2 \| \, \| e_{p_{k,1}} - e_{p_{k,2}} \|_k^{-1} \| \cdot \|_k. \end{aligned}$$

Then if $n_k > m$, $U_k(\Sigma_1^m x_i e_i) = \Sigma_1^m x_i e_{p_{k,i}} - (n_k - 1)(\Sigma_1^m x_i) e_{p_{k,n_k}}$.

Since $(n_k - 1)^{-1} \Sigma_{i=1}^m x_i \to 0$ as $k \to \infty$ and $\| e_{p_{k,n_k}} \|_k \leqq \| e_{p_{k,n_k}} \| = \| e_1 \|$, using the fact that $\delta(\{e_i\} \subset E) = 1$ it follows that

$$\begin{aligned} \left\| \sum_{i=1}^{m} x_i e_i \right\| \| e_1 - e_2 \|^{-1} &= \lim_{k \to \infty} \left\| U_k\left(\sum_{i=1}^{m} x_i e_i\right) \right\| \| e_1 - e_2 \|^{-1} \\ &= \lim_{k \to \infty} \left\| \sum_{i=1}^{m} x_i e_{p_{k,i}} \right\|_k \| e_{p_{k,1}} - e_{p_{k,2}} \|_k^{-1}. \end{aligned}$$

Therefore for any vector $x = \Sigma_{i=1}^{\infty} x_i e_i$ the following equality holds

$$\| x \| \, \| e_1 - e_2 \|^{-1} = \lim_{m \to \infty} \lim_{k \to \infty} \left\| \sum_{i=1}^{m} x_i e_{p_{k,i}} \right\|_k \| e_{p_{k,1}} - e_{p_{k,2}} \|_k^{-1},$$

thus right hand side defines a norm on E, necessarily a Hilbert norm, contradicting the fact that E is not isometric to l_2. □

§2. G is finite

The group of all permutations of $\{1, 2, \cdots, n\} = [1, n]$ is written S_n, and $\Delta_n = \{-1, 1\}^n$ is the group of all n-tuples of signs. The main result of this section is the following:

THEOREM 2.1. *Let E be a real n-dimensional normed space, $n \geqq 13$, with $\delta(E)=1$ and with G, the group of isometries of E, finite. One and only one of the following statements is true*:

(1) *There is a basis $\{b_i, b_i'\}_{i\leqq n}$ for E such that G is the group of operators $\varepsilon \sum_{i=1}^n b_i' \otimes b_{\Pi(i)}$, $\Pi \in S_n$ and $|\varepsilon|=1$.*

(2) *There is a basis $\{b_i, b_i'\}_{i\leqq n}$ for E such that G is the group generated by the operators $\pm \sum_{i=1}^n b_i' \otimes b_{\Pi(i)}$ and the operator $1_E - (2/n)\,(\sum_{i=1}^n b_i') \otimes (\sum_{i=1}^n b_i)$.*

(3) *There is a basis $\{b_i, b_i'\}_{i\leqq n}$ for E such that G is the group of operators $\sum_{i=1}^n \varepsilon_i b_i' \otimes b_{\Pi(i)}$, $\Pi \in S_n$ and $(\varepsilon_i) \in \Delta_n$.*

(4) *n is even and there is a basis $\{b_i, b_i'\}_{i\leqq n}$ for E with G the group of operators $\sum_{i=1}^n \varepsilon_i b_i' \otimes b_{\Pi(i)}$, $\Pi \in S_n$ and $(\varepsilon_i) \in \Delta_n$ satisfying $\Pi_{i=1}^n \varepsilon_i = 1$.*

(5) *There is an $(n+1)$-dimensional space F with a monotone diagonally symmetric basis $(f_i, f_i'\}_{i\leqq n+1}$ and an isometry $V: E \to F$ such that $V(E)$ is the kernel of $\sum_{i=1}^{n+1} f_i'$, and such that G is the group of operators $\varepsilon V^{-1}(\sum_{i=1}^n f_i' \otimes f_{\Pi(i)})V$, $\Pi \in S_{n+1}$ and $|\varepsilon|=1$.*

The proof will be broken down into a series of Lemmas. We retain the notations of Section 1, and adopt the following convention: If $x, y \in E$, $x \otimes y$ will stand for the operator in $L(E)$ defined by $x \otimes y(z) = (x,z)y$. However, if $x' \in E'$ and $x \in E$, $x' \otimes x$ is defined by $x' \otimes x(y) = \langle y, x'\rangle x$.

A point $x \in E$ is called a reflection point if $\|x\|_2 = 1$ and if the operator $1_E - 2x \otimes x$ is an isometry of E. The set of all reflection points is denoted by R. Observe that if $x \in R$ and $g \in G$ then $g(x) \in R$, since $\|g(x)\|_2 = \|x\|_2 = 1$ and $1_E - 2g(x) \otimes g(x) = g(1_E - 2x \otimes x)g^*$.

From this point on we shall always assume, unless stated otherwise, that G is finite. The following Lemma was used implicitly without proof in [7].

LEMMA 2.2. *Let $z \in E$ and $x_i \in R$, $i = 1, 2$, both satisfy*

$$0 < |(x_i, z)| = \inf\{|(y,z)|;\ y \in R \text{ and } (y,z) \neq 0\}.$$

(1) *If $y \in R$ and $(y,z)=0$, then $|(x_1, y)| = \cos(\pi/m)$ for some integer $m \geqq 2$.*

(2) *For some positive integer q, $|(x_1, x_2)| = \cos(2\pi/q)$.*

(3) *If $z \in R$ then $|(x_2, z)| = \sin(\pi/m)$ for some positive even integer $m \geqq 2$.*

PROOF. (1) We may assume $\|z\|_2$ and choose an orthonormal basis $(b_i)_{i\leqq n}$ for R^n such that $y = b_1$, $x_1 = \cos(\beta/2)b_1 + \sin(\beta/2)b_2$, $z = (\cos\alpha)b_2 + (\sin\alpha)b_3$. Let $z_k = \cos(k\beta/2)b_1 + \sin(k\beta/2)b_2$ $(k = 0, \pm 1, \pm 2, \cdots)$.

The operators $g_2 = 1_E - 2y \otimes y$ and $g_1 = 1_E - 2x_1 \otimes x_1$ are in G, so are therefore also the operators

$$(g_2g_1)^k = (1_E - 2y \otimes y)(1_E - 2z_k \otimes z_k) \text{ and } g_2(g_2g_1)^k = 1_E - 2z_k \otimes z_k;$$

the above formulas are easily established by mathematical inductions. Hence $z_k \in R$. Since G is finite, $(g_2g_1)^m = 1_E$ for some integer $m \geqq 1$; this implies $\cos(m\beta) = 1$ so $\beta = 2\pi p/m$ for some integer p.

Let $d = (m,p)$ be the greatest common divisor of m and p. If $d > 1$, then $(g_2g_1)^{m/d} = 1_E$ so we may assume $d = 1$, then $sp = tm + 1$ for some integers s and t.

Since $z_k \in R$, $|(z_k, z)| \geqq |(x_1, z)| > 0$ whenever $(z_k, z) \neq 0$, this is equivalent to the inequality $|\sin(kp\pi/m)| \geqq |\sin(p\pi/m)|$ whenever left hand side is not zero. Taking $k = ls$ we get $|\sin(l\pi/m)| \geqq |\sin(p\pi/m)|$ whenever left hand side is not zero. Therefore $p = 1$ and $m \geqq 2$.

(2) Assume again $\|z\|_2 = 1$ and choose an orthonormal basis such that $x_1 = \cos(\beta/2)b_1 + \sin(\beta/2)b_2$, $x_2 = b_1$, $z = \cos\theta\cos\phi b_1 + \cos\theta\sin\phi b_2 + \sin\theta b_3$.

Let $g_1 = 1_E - 2x_1 \otimes x_1$, $g_2 = 1_E - 2x_2 \otimes x_2$.

Defining z_k, m, p, s, t as in (1) we obtain for every integer k

$$|(z_k, z)| \geqq |(x_1, z)| = |(x_2, z)| > 0 \quad \text{whenever} \quad (z_k, z) \neq 0,$$

or equivalently

$$\left|\cos\left(\phi - \frac{kp\pi}{m}\right)\right| \geqq |\cos(\phi - p\pi/m)| = |\cos\phi|$$

$$> 0 \quad \text{whenever} \quad \cos\left(\phi - \frac{kp\pi}{m}\right) \neq 0.$$

Taking $k = sl$ ($l = 0, \pm 1, \pm 2, \cdots$), we get

$$|\cos(\phi - l\pi/m)| \geqq |\cos(\phi - p\pi/m)| = |\cos\phi|$$

whenever $\cos(\phi - l\pi/m) \neq 0$. This implies $p = 1$ or 2, therefore $\beta/2 = 2\pi/q$ for some positive integer q.

(3) Assume $\|z\|_2 = 1$ and choose an orthonormal basis such that $x_2 = b_1$, $z = \cos(\beta/2)b_1 + \sin(\beta/2)b_2$. Let $g_1 = 1_E - 2z \otimes z$ and $g_2 = 1_E - 2x_2 \otimes x_2$. Defining z_k, m, p, s, t as in (1), then for any integer k, $|(z_k, z)| \geqq |(x_2, z)| > 0$ whenever $(z_k, z) \neq 0$. Taking $k = ls$, $|\cos(l\pi/m)| \geqq |\cos(p\pi/m)|$ whenever $\cos(l\pi/m) \neq 0$. The last inequality implies either $pm^{-1} - 1/2$ is an integer or $(p + \varepsilon)m^{-1} - 1/2$ is an integer for some $|\varepsilon| = 1$. Hence m is an even integer.

It follows that $|(x_2, z)| = |\cos(p\pi/m)| = \sin(\pi/m)$ for $m \geqq 2$. □

Lemma 2.3. *Let $(b_i)_{i \leqq n}$ be an orthonormal basis for E and suppose $x \in E$ satisfies $\|x\|_2 = 1$,*

$(x, b_i) = a$ *for* $i = k+1, k+2, \cdots, k+m$, *and*

$(x, b_i) = a + \varepsilon 2^{1/2}\cos(\pi/s)$ *for* $i = k+m+1, \cdots, n$,

where $0 \leqq k \leqq n-10$, $1 \leqq m \leqq n-k-1$, $|\varepsilon| = 1$ *and* $s \geqq 3$ *are integers and* a *is a constant. Then either* $s = 3$ *and* $m = 1, 2, n-k-2$ *or* $n-k-1$; *or* $s = 4$ *and* $m = 1$ *or* $n-k-1$.

PROOF. The discriminant of the quadratic in a obtained by expanding $\|x\|_2^2 = 1$ is non-negative, so

$$0 \leqq (n-k)\left[1 - \sum_{i=1}^{k} |(x, b_i)|^2\right] - 2m(n-m-k)\cos^2(\pi/s)$$

$$\leqq (n-k) - 2m(n-m-k)\cos^2(\pi/s)$$

and thus

$$(*) \qquad 2\mu m(n-m-k) \leqq n-k$$

where $\mu = \cos^2(\pi/s)$. Observe that $f(t) = k + t + t(2\mu t - 1)^{-1}$ is increasing on $[\mu^{-1}, \infty)$ and that $g(t) = k + 1 + (2t-1)^{-1}$ is decreasing on $(2^{-1}, \infty)$.

To obtain a contradiction suppose $s \geqq 5$. Then $(*)$ gives $n \leqq f(m)$. If $m = 1$ then $n \leqq f(1) = g(\mu)$. In case $m \geqq 2$, $n \leqq f(m) \leqq f(n-k-1)$ since f is increasing on $[2, \infty)$, which implies $n \leqq g(\mu)$. In either case

$$n \leqq g(\mu) \leqq g(\cos^2(\pi/5)) = k + 2 + 5^{1/2},$$

contradicting $k + 10 \leqq n$.

If $s = 3$ and $3 \leqq m \leqq n-k-3$, $(*)$ gives $n \leqq f(m)$. If $m = 3$, then $n \leqq f(3)$ implies $n \leqq k+9$, a contradiction. In case $m \geqq 4$, $n \leqq f(m) \leqq f(n-k-3)$ since f is increasing on $[4, \infty)$, which implies again $n \leqq 9 + k$.

If $s = 4$ and $2 \leqq m \leqq n-k-2$ then $(*)$ again yields $n \leqq f(m) \leqq f(n-k-2)$ the last inequality since f is increasing on $[2, \infty)$, which implies $n \leqq k+4$, a contradiction. □

The next elementary result will be used without proof.

LEMMA 2.4. *Let* s *and* t *be positive integers with* $\operatorname{card}\{s, t, 2\} = 3$, *and let* $|\varepsilon| = 1$. *The equation* $|\cos(\pi/s) + \varepsilon\cos(\pi/t)| = \cos(\pi/r)$ *is not satisfied for any integer* r.

LEMMA 2.5. *Let* $B = \{b_i\}_{i \leqq n}$ *be an orthonormal basis for* E *with* $\delta(B) = 1$. *Let* $x \in R$, $y \in E$, $0 \leqq k \leqq n-10$, *be such that*

$$|(x, y)| = \inf\{|(z, y)|;\ z \in R \text{ and } (z, y) \neq 0\}$$

and $(y, b_i - b_j) = 0$ for all integers i and j satisfying $k < i, j \leqq n$. Then there exists a permutation $\Pi \in S_n$ satisfying $\Pi(i) = i$ for all $i = 1, 2, \cdots, k$ and integers m, s with $1 \leqq m < n - k$ and $2 \leqq s \leqq 4$, $|\varepsilon| = 1$ such that if $u = \Sigma_{i=1}^n b_i \otimes b_{\Pi(i)}$

$$(u(x), b_i) = a \quad for \quad i = k+1, k+2, \cdots, m+k,$$

$$(u(x), b_i) = a + \varepsilon 2^{1/2} \cos(\pi / s) \quad for \quad i = m+k+1, \cdots, n,$$

where a is a fixed scalar. Moreover, if $s \neq 2$, then either $s = 3$ and $m = 1$, 2, $n-k-2$ or $n-k-1$; or, $s = 4$ and $m = 1$ or $n-k-1$.

Proof. Since $2^{1/2}(b_i - b_j) \in R$ if $i \neq j$ and orthogonal to y for $i, j > k$, it follows by Lemma 2.2(1) that $(x, 2^{-1/2}(b_i - b_j)) = \varepsilon_{ij} \cos(\pi / m_{ij})$ where $|\varepsilon_{ij}| = 1$, $m_{ij} \geqq 2$ are integers and $i \neq j > k$. Hence

$$|\cos(\pi / m_{ij}) - \varepsilon_{ij}\varepsilon_{il} \cos(\pi / m_{il})| = \cos(\pi / m_{jl}),$$

but by Lemma 2.4 card$\{m_{ij}, m_{il}, 2\} \leqq 2$ in all cases, therefore either $m_{ij} = 2$ or $m_{ij} = s$ where $s \geqq 2$ is a fixed integer. Since $|\cos(\pi/s) - \sigma \cos(\pi/s)| = \cos(\pi/t)$ for $|\sigma| = 1$ and $s > 2$ implies $\sigma = 1$ and $t = 2$ we may assume $\varepsilon_{ij} = \varepsilon$ in all cases. Thus after a suitable permutation Π (if necessary) we get that $(u(x), b_i) = a$ for $k+1 \leqq i \leqq k+m$ and $(u(x), b_i) = a + \varepsilon 2^{1/2} \cos(\pi/s)$ for $k+m < i \leqq n$. If $s \neq 2$ the last statement follows from Lemma 2.3. □

Lemma 2.6. *If $n = \dim(E) \geqq 10$ and $\delta(E) = 1$, then E has an orthonormal basis $B = \{b_i\}_{i \leqq n}$ with $\delta(B) = 1$ which satisfies one of the following:*

(a) *Every reflection point not equal to $\pm n^{-1/2} b$, where $b = \Sigma_{i=1}^n b_i$, is in $[b]^\perp$.*

(b) *For $1 \leqq k \leqq n$, $b_k \in R$.*

(c) *For $1 \leqq i$, $k \leqq n$, $i \neq k$, $2^{-1/2}(b_i + b_k) \in R$.*

(d) *For μ a suitable constant, $2^{-1/2}(b_k + \mu b) \in R$, $1 \leqq k \leqq n$. Further, in case* (d) *$n^{-1/2} b \notin R$.*

Proof. By Lemma 1.1 E has an orthonormal basis $B = \{b_i\}_{i \leqq n}$ with $\delta(B) = 1$. Write $b = \Sigma_{i=1}^n b_i$. The desired basis for E will be either B or $\{u(b_i)\}_{i \leqq n}$ where $u = 1_E - (2/n) b \otimes b$. Observe that this last basis is also orthonormal and diagonally symmetric since u commutes with each permutation of the b_i's.

Suppose (a) is not the case, and let $x \in R$ satisfy

$$|(x, b)| = \inf\{|(z, b)|;\ z \in R \quad \text{and} \quad (z, b) \neq 0\}.$$

Then $(x, b) \neq 0$ and $x \neq \pm n^{-1/2} b$. The reflexion x may be replaced by any vector obtained by the action of a permutation on x, so applying Lemma 2.5 with $y = b$ shows that we may assume $(x, b_k) = a$ for $1 \leqq k \leqq m$ and $(x, b_k) =$

$a + \varepsilon 2^{1/2}\cos(\pi/s)$ for $m < k \leqq n$ where $|\varepsilon| = 1$ and $2 \leqq s \leqq 4$ is a fixed integer. $s \neq 2$ since otherwise $x = \pm n^{-1/2}b$. Hence by Lemma 2.4, as $n \geqq 10$, either $s = 3$ and $m = 1, 2$, $n-2$ or $n-1$; or, $s = 4$ and $m = 1$ or $n-1$.

If $s = 4$, expanding $\|x\|_2^2$ as a quadratic in a and solving yields a sign σ with $a + \varepsilon = (\varepsilon + \sigma)n^{-1}$ if $m = 1$, and $a = (\sigma - \varepsilon)n^{-1}$ if $m = n - 1$. In either case x has form $\pm b_k$ or $\pm u(b_k)$, $k = 1$ or n. But if some $\pm b_k \in R$, then $b_i \in R$ for all $i = 1, 2, \cdots, n$, and similarly for the $u(b_i)$'s since u commutes with each permutation of the b_i's.

The other cases may be handled similarly. For $s = 3$ and $m = 1$ or $n - 1$, $x = \pm 2^{-1/2}(b_k + \mu b)$, $k = 1$ or n. For $s = 3$ and $m = 2$ or $n - 2$, x is $\pm 2^{-1/2}(b_i + b_k)$, $i \neq k$, or the image of such a vector by u. Thus (c) and (d) are also possibilities.

If (d) holds the proof above shows that $x = 2^{-1/2}(b_1 + \mu b) \in R$ satisfies

$$|(x, n^{-1/2}b)| = \inf\{(y, n^{-1/2}b);\, y \in R \quad \text{and} \quad (y, b) \neq 0\}.$$

Thus if $n^{-1/2}b \in R$, Lemma 2.2(3) implies that there is an integer r such that $\cos(2\pi/r) = 1 - n^{-1}(1 + n\mu)^2 = -n^{-1}$, the last since $\|x\|_2 = 1$. This is impossible since $n \geqq 10$. □

In the following lemmas $B = \{b_i\}_{i \leqq n}$ represents the orthonormal basis for E with $\delta(B) = 1$, $b = \Sigma_{i=1}^n b_i$.

LEMMA 2.7. *Let $y \in R$.*

(1) *If* (c) *of Lemma* 2.6 *holds for B and if $n \geqq 12$, then $(y, b_k) = 0$ or $|(y, b_k)| \geqq 2^{-1/2}$ for each $k = 1, 2, \cdots, n$.*

(2) *If* (a) *of Lemma* 2.6 *holds for B and if $n \geqq 11$ then $(y, b_k) = 0$ or $|(y, b_k)| \geqq n^{-1/2}$ for each $k = 1, 2, \cdots, n$.*

PROOF. It is enough to establish (1) and (2) for $k = 1$. To this end denote by $x = (x_1, \cdots, x_n)$ the vector $\Sigma_{k=1}^n x_k b_k$, and assume $x \in R$ and satisfies

$$|x_1| = \inf\{|(z, b_1)|;\, z \in R \quad \text{and} \quad (z, b_1) \neq 0\}.$$

Then $0 < |x_1| \leqq |(2^{-1/2}(b_1 - b_2), b_1)| = 2^{-1/2}$.

By Lemma 2.5 we may suppose there is a constant a, a sign ε and positive integers m and $s(\geqq 2)$ such that $x_i = a$ for $2 \leqq i \leqq m + 1$, $x_i = a + \varepsilon 2^{1/2}(\cos(\pi/s))$ for $1 + m < i \leqq n$.

If $s = 2$ then $m = n - 1$. If $s \geqq 3$, by Lemma 2.5 either $s = 3$ and $m = 1, 2$, $n - 3$ or $n - 2$; or, $s = 4$ and $m = 1$ or $n - 2$.

Assume B satisfies case (c). If $s = 2$ then $1 = \|x\|_2^2 = x_1^2 + (n-1)a^2$, and by Lemma 2.2(1)

$$2^{1/2}|a| = |(x, 2^{-1/2}(b_2 + b_3))| = \cos(\pi/t)$$

for some integer $t \geqq 2$, therefore $\cos(\pi/t) \leqq \sqrt{2/(n-1)}$, so $t = 2$, $a = 0$, $|x_1| = 1$ — contradicting $|x_1| \leqq 2^{-1/2}$.

If $s = 4$, either $x = (x_1, a, \underbrace{a+\varepsilon, \cdots, a+\varepsilon}_{n-2})$ or $x = (x_1, \underbrace{a, \cdots, a}_{n-2}, a+\varepsilon)$. By symmetry, sufficient to consider the second case. Since $1 \geqq (n-2)a^2$, we shall obtain as above $a = 0$, so $x_1 = 0$, a contradiction.

Thus $s = 3$ and by symmetry again it is sufficient to consider the cases $m = n - 3$ and $m = n - 2$. As above, the case $m = n - 3$ implies $a = x_1 = 0$, so is impossible. If $m = n - 2$, $x = (x_1, a, \cdots, a, a + \varepsilon 2^{-1/2})$ establishing again $a = 0$, so $|x_1| = 2^{-1/2}$. Thus if $y \in R$, either $y_1 = 0$ or $|y_1| \geqq |x_1| = 2^{-1/2}$; this is true for all coordinates of y, establishing the result.

Assume B satisfies (a). Case (2) is established if $x = \pm n^{-1/2} b$, so assume $(x, b) = 0$. Expanding $\|x\|_2^2 = 1$ and using $(x, b) = 0$ shows after elimination of a that $nx_1^2 = n - 1 - 2m(n - m - 1)\cos^2(\pi/s)$.

In each of the seven cases listed above the equality yields $nx_1^2 \geqq 1$, establishing the assertion. □

LEMMA 2.8. *Let $y \in R$. If* (a) *of Lemma* 2.6 *holds for B and if $n \geqq 13$, then* $(y, b_i - b_k) = 0$ *or* $|(y, b_i - b_k)| \geqq 2^{-1/2}$ *for all $i \neq k$.*

PROOF. It is enough to give the proof for $b_1 - b_2$. Let $x \in R$ satisfy

$$|(x, b_1 - b_2)| = \inf\{|(z, b_1 - b_2)|;\ z \in R \quad \text{and} \quad (z, b_1 - b_2) \neq 0\}.$$

Then $x \neq n^{-1/2} b$, so $(x, b) = 0$. Since $n \geqq 12$ we obtain (permuting the coordinates of x if necessary) that $(x, b_k) = a$ for $3 \leqq k \leqq m + 2$, $(x, b_k) = a + \varepsilon 2^{1/2}\cos(\pi/s)$ for $m + 3 \leqq k \leqq n$, where a is a constant, $|\varepsilon| = 1$ and m and s satisfy: $s = 2$ and $m = n - 2$; or, $s = 3$ and $m = 1, 2, n - 4$ or $n - 3$; or $s = 4$ and $m = 1$ or $n - 3$.

There are three cases to consider.

Case 1. Both a and $a + \varepsilon 2^{1/2}\cos \pi/s$ are zero. The equalities $\|x\|_2 = 1$, and $(x, b) = 0$ imply here that $x = \pm 2^{-1/2}(b_1 - b_2)$, which is impossible since

$$|(x, b_1 - b_2)| \leqq |(2^{-1/2}(b_2 - b_3), b_1 - b_2)| = 2^{-1/2}.$$

Case 2. Exactly one of a, $a + \varepsilon 2^{1/2}\cos(\pi/s)$ is zero. If for instance $a = 0$, checking the seven·possibilities together with $\|x\|_2 = 1$, $(x, b) = 0$, shows that $s = 3$, $m = n - 3$ and $|(x, b_1 - b_2)| = 2^{-1/2}$.

Case 3. Neither a nor $a + \varepsilon 2^{1/2}\cos(\pi/s)$ is zero. We shall see this is impossible. Lemma 2.7(2) implies that $|a|$, $|a + \varepsilon 2^{1/2}\cos(\pi/s)| \geqq n^{-1/2}$, therefore $x_1^2+x_2^2 \leqq 2n^{-1}$, where $x_i=(x,b_i)$, and hence $(x_1 \pm x_2)^2 \leqq 4n^{-1}$. s cannot be equal to 2, since otherwise $x_1+x_2+(n-2)a=0$, so $|a| \leqq 2n^{-1/2}(n-2)^{-1}$ and then $1=\|x\|_2^2=x_1^2+x_2^2+(n-2)a^2<1$, which is a contradiction.

To eliminate the cases $s=3$ and 4, let $z \in R$ be the point obtained by permuting in x the $m+2^{\text{nd}}$ with the $m+3^{\text{rd}}$ coordinate and the 1st with the 2nd coordinate. By Lemma 2.2(2), using the fact that

$$1=\|x\|_2^2=x_1^2+x_2^2+ma^2+(n-m-2)(a+\varepsilon 2^{1/2}\cos(\pi/s))^2,$$

we get for some positive integer t

$$|1-(x_1-x_2)^2-2\cos^2(\pi/s)|=|(x,z)|=\cos(2\pi/t).$$

For $s=4$, $|\cos(\pi/t)|=(x_1-x_2)^2 \leqq 4n^{-1}<\cos(2/5)$ since $n \geqq 13$, so $t=4$ and hence $x_1-x_2=0$, a contradiction.

For $s=3$, $1/2 \geqq 1/2-(x_1-x_2)^2 \geqq (n-8)(2n)^{-1}>\cos(2\pi/5)$, so $t=6$ and again $x_1-x_2=0$. □

LEMMA 2.9. *Let $x \in R$. (1) If B satisfies* (a) *of Lemma* 2.6 *and $n \geqq 13$, then $x=\pm n^{-1/2}b$ or $x=2^{-1/2}(b_i-b_k)$ for some $i \neq k$. (2) If B satisfies* (c) *of Lemma* 2.6 *and $n \geqq 12$, then $x=\pm b_k$ or $\pm 2^{-1/2}(b_i \pm \varepsilon b_k)$ for some i and k, $i \neq k$, $|\varepsilon|=1$.*

PROOF. Part (2) follows immediately by Lemma 2.7(1). Assume B satisfies (a) and let $x \in R$, $(x,b)=0$. If $(x,b_i) \neq 0$ for all i then by Lemma 2.7(2), $|(x,b_i)|=n^{-1/2}$ for all i. Since $(x,b)=0$, $(x,b_i-b_k)=2n^{-1/2}$ for some $i \neq k$, contradicting Lemma 2.8. Thus $(x,b_k)=0$ for some k, so $(x,b_i)=0$ or $|(x,b_i)| \geqq 2^{-1/2}$ for any i, by Lemma 2.8. This with $\|x\|_2=1$ establishes (1). □

LEMMA 2.10. *Suppose $n \geqq 10$ and that $g \in G$ maps each vector b_i-b_k, $i \neq k$ to a vector of the form $\sigma(b_s+\varepsilon b_t)$, $|\sigma|=|\varepsilon|=1$ and $s \neq t$. Then there is a $\Pi \in S_n$ and $(\varepsilon_i) \in \Delta_n$ such that g equals $\Sigma_{i=1}^n \varepsilon_i b_i' \otimes b_{\Pi(i)}$ or $u\Sigma_{i=1}^n \varepsilon_i b_i' \otimes b_{\Pi(i)}$, where $b_i' \in E'$ are the coefficient functionals corresponding to the b_i, and $u=1-(2/n)b' \otimes b$, $b'=\Sigma_{i=1}^n b_i'$.*

PROOF. For $1 \leqq s \leqq n$ and $i \neq k$, $(b_i-b_k,\ g^*(b_s))=0,1$ or -1, so that $g^*(b_s)=\pm n^{-1/2}b$, or there is a constant a, a sign ε and an integer $m<n$ with $g^*(b_s)$ a permutation of $z=a\Sigma_{i=1}^m b_i+(a+\varepsilon)\Sigma_{i>m}b_i$. In the latter case Lemma 2.3 implies $m=1$ or $m=n-1$. Using $\|x\|_2=1$ to solve for a, $z=\pm b_k$ or $\pm u(b_k)$, $k=1$ or n. The operator g^* is orthogonal so either each $g^*(b_s)$ is $A \pm b_r$, or each $g^*(b_s)$ is $A \pm u(b_r)$. This is enough to establish the desired representation of g. □

PROOF OF THEOREM 2.1. Let $B = \{b_i, b'_i\}_{i \leq n}$ be any orthonormal basis for B, $\delta(B) = 1$, which satisfies one of (a) through (d) of Lemma 2.6.

Assume (b) holds. Since $2^{-1/2}(b_i - b_k) \in R$ for $i \neq k$, it follows (c) must hold, then by Lemma 2.9 R consists of the points $\pm b_k$ and $2^{-1/2}(\varepsilon b_i + \delta b_k)$, for all $|\varepsilon| = |\delta| = 1$, $1 \leq i \neq k \leq n$. Since any $g \in G$ maps R onto R, G must satisfy (3) of Theorem 2.1.

Assume (c) holds and no b_k is a reflection point, then (4) must follow. To see this notice first that n is even, for if $n = 2m + 1$ is odd, then $1_E - 2b'_n \otimes b_n = -\Pi_{i=1}^m g_i h_i$, where g_i and h_i are the reflections associated with $2^{-1/2}(b_{2i} - b_{2i-1})$ and $2^{-1/2}(b_{2i} + b_{2i-1})$, so $b_n \in R$, contrary to the assumption. Next, $u = 1_E - (2/n)\, b' \otimes b$ is not in G, since by Lemma 2.9(2) $2^{-1/2}u(b_1 + b_2) \notin R$. By Lemma 2.9(2) and 2.10 each $g \in G$ has the form $\Sigma_{i=1}^n \varepsilon_i b'_i \otimes b_{\Pi(i)}$. Then $\Pi_1^n \varepsilon_i = 1$, since if not, arguing as above would show that some $b_k \in R$.

Of course every $\Sigma_{i=1}^n \varepsilon_i b'_i \otimes b_{\Pi(i)}$ ($\Pi \in S_n$, $(\varepsilon_i) \in \Delta_n$, $\Pi_1^n \varepsilon_i = 1$) is in G, since it is the product of permutations and operators of the form

$$[1_E - (b'_i - b'_k) \otimes (b_i - b_k)]\,[1_E - (b'_i + b'_k) \otimes (b_i + b_k)].$$

Now assume (a) holds for B, and let $g \in G$. By Lemma 2.9(1) for every $i \neq k$, $g(2^{-1/2}(b_i - b_k))$ is either of the form $2^{-1/2}(b_r - b_s)$ or $\pm n^{-1/2} b$. If for some $i \neq k$, $g(b_i - b_k) = \sqrt{2/n}\, b$, then for any $j \neq i$ and k, $g(b_i - b_j) = b_r - b_s$, so that $g(b_j - b_k)$ does not have the required form. Hence g satisfies the conditions of Lemma 2.10 so $g(b_1 - b_k) = \varepsilon_1 b_{\Pi(1)} - \varepsilon_k b_{\Pi(k)}$ for every $k \neq 1$, hence $\varepsilon_1 = \varepsilon_k$ and g has the form in (2) or (1) depending on whether or not $u \in G$.

If case (d) is satisfied then $2^{-1/2}(b_k + \mu b) \in R$, $1 \leq k \leq n$. Write $e_k = b_k + \mu b$ and $e'_k = b'_k + \nu b'$, where ν is chosen to satisfy $0 = \nu + \mu + n\mu\nu$ (possible, since $\mu \neq -n^{-1}$). Since $\langle e_k, e'_i\rangle = \delta_{ik}$, $\{e_i\}_{i \leq n}$ is a basis for E with $\delta(\{e_i\}) = 1$. Let $g_\Pi = \Sigma_{i=1}^n e'_i \otimes e_{\Pi(i)}$, $\Pi \in S_n$, then $g_\Pi \in G$ and $u_k = 1_E - (e'_k + e') \otimes e_k$, $e' = \Sigma_1^n e'_k$, is the reflection about $2^{-1/2}(b_k + \mu b)$, therefore $u_k \in G$.

Let $\{f_i, f'_i\}_{i \leq n}$ be the unit vector basis for R^{n+1}, $f = \Sigma_1^n f_i$, $f' = \Sigma_1^n f'_i$ and define $v : E \to R^{n+1}$ by $v(e_k) = f_k - f_{n+1}$, $1 \leq k \leq n$. For $\tau \in S_{n+1}$ write $h_\tau = \Sigma_1^{n+1} f'_i \otimes f_{\tau(i)}$. Observe that $v^{-1} h_\tau v = g_{\tau|[1,n]}$ if τ fixes $n+1$, and that $v^{-1} h_{\tau_k} v = u_k$ if τ_k is the 2-cycle interchanging $n+1$ with k, $1 \leq k \leq n$. Thus $v^{-1} h_\tau v \in G$ for all $\tau \in S_{n+1}$.

Now let F be R^{n+1} normed by

$$|z| = |\langle z, f'\rangle| + \| v^{-1}(1_E - P)(z)\|$$

where $P = (n+1)^{-1} f' \otimes f$. Clearly v is an into isometry whose range is the kernel of f', and h_τ is an isometry of F for each $\tau \in S_{n+1}$, since

$$\| v^{-1}(1_E - P)h_\tau z \| = \| v^{-1}h_\tau^{-1} v v^{-1}(1_E - P)h_\tau z \| = \| v^{-1}(1_E - P)z \|$$

so $\delta(\{f_i\}_{i \leqq n+1}) = 1$. Notice that if $h \in G$ and $|\varepsilon| = 1$, then $\tilde{h} = vh v^{-1}(1_E - P) + \varepsilon P$ is an isometry of F. Taking $h = 1_E$, a scalar multiple of f is a reflection point in F. The basis $w_i = \alpha(f_i + \beta f)$, $1 \leqq i \leqq n+1$, is orthonormal for suitable α and β, and reflection about $n^{-1/2}w$ ($w = \Sigma_1^{n+1} w_i$) is an isometry, so $\{w_i\}_{i \leqq n+1}$ must satisfy one of the conclusions of Lemma 2.6. Since $n^{-1/2}w$ is a reflection point, (d) is impossible, and (b) and (c) are also impossible by the previously proven parts of the Theorem. Thus $\{w_i\}_{i \leqq n+1}$ must satisfy (a), and so for $h \in G$, $\tilde{h}$ has form εh_τ or $\varepsilon u h_\tau$, u is the reflection about $n^{-1/2}w$, h_τ is a permutation of the w_i's. But each permutation h_τ of w_i's is a permutation of the f_i's and u is the identity on $v(E)$, so $h \pm v^{-1}h_\tau v$. This proves (5) must hold, and concludes the proof of the Theorem. □

Remarks. In each of the cases (1)–(5) the basis B can be taken to be orthonormal. The number of isometries in case (1) is $2(n!)$, in case (2) is $4(n!)$, in case (3) is $2^n n!$, in case (4) is $2^{n-1}n!$, and in case (5) is $2((n+1)!)$.

The argument given in proof establishes also the following:

Corollary 2.11. *Let* $\dim(E) = n \geqq 13$, *E have a basis* $\{e_i\}_{i \leqq n}$ *with* $\delta(\{e_i\}) = 1$ *and* $\Sigma_1^n e_i$ *a scalar multiple of a reflection point. If the group of isometries of E is finite, then it satisfies* (2) *of Theorem* 2.1.

Proof. By the proof given above case (5) of the Theorem arises only from case (d) of Lemma 2.6, which is not satisfied here, therefore (5) does not hold here, and clearly (2) must follow. □

Remark. Let $n = \dim(E) \geqq 13$ with $\delta(E) = 1$ and G finite. The proof of Theorem 2.1 yields an orthonormal basis for E such that the set of reflection points of E consists of the vectors contained in exactly one of the following 5 cases:

(1) the vectors $2^{-1/2}(b_i - b_k)$, $i \neq k$;

(2) the vectors $2^{-1/2}(b_i - b_k)$, $i \neq k$, and $\pm n^{-1/2}b$;

(3) the vectors $\pm b_k$ and $\pm 2^{-1/2}(b_i + \varepsilon b_k)$, $i \neq k$, $|\varepsilon| = 1$;

(4) the vectors $\pm 2^{-1/2}(b_i + \varepsilon b_k)$, $i \neq k$, $|\varepsilon| = 1$, and where n is even;

(5) the vectors $2^{-1/2}(b_i - b_k)$, $i \neq k$, together with $\pm 2^{-1/2}(b_i + \mu b)$, where μ is one of the two possible solutions to $(n\mu + 1)^2 = n + 1$.

Observe also that in Lemma 2.6 only one of cases (a), (c) or (d) may hold, and that (b)$\Rightarrow$(c) and (b) holds iff $t(E) = 1$ iff (3) of Theorem 2.1 holds.

COROLLARY 2.12. *Let* $\dim(E) = n \geqq 13$, $\delta(E) = 1$ *and* G, *the group of isometries of* E, *be finite.* G *satisfies* (3) *or* (4) *of Theorem* 2.1 *iff* $\chi(E) = 1$. G *satisfies* (3), (4) *or* (5) *of Theorem* 2.1 *iff* G *has trivial commutator.*

PROOF. Let $\{b_i, b_i'\}_{i \leq n}$ be an orthonormal basis for E so that R can be described as in the preceeding remark.

If G satisfies (4) then $n = 2m$ is even and the basis

$$e_i = 2^{-1/2}(b_{2i} + b_{2i-1}),\ e_{m+i} = 2^{-1/2}(b_{2i} - b_{2i-1}),\ i = 1, 2, \cdots, m$$

clearly satisfies $\chi(\{e_i\}) = 1$. If G satisfies (3), then $\chi(B) = 1$.

To obtain a contradiction suppose G satisfies (1), (2) or (5) and that $x(E) = 1$. Let $\{e_i\}_{i \leq n}$ be a basis with $\chi(\{e_i\}) = 1$. Such a basis must necessarily be orthogonal. Since $\| e_i \|_2^{-1} e_i \in R$ it is of the form $\pm n^{-1/2} b$, $\pm 2^{-1/2}(b_r + \mu b)$ or $2^{-1/2}(b_r - b_s)$. As the e_i's are orthogonal at most two can be of the first two types, but then there can be no $n-2$ orthogonal vectors of the third type, a contradiction.

If G satisfies (1) or (2) then $b' \otimes b$ commutes with each isometry of E, so E does not have a trivial commutator. Clearly G has trivial commutator if G satisfies (3).

If G satisfies (4), and v commutes with each $g \in G$, it must have the form $\lambda 1_E + \mu b' \otimes b$ since it commutes with all permutations. Let $\varepsilon_1 = \varepsilon_2 = -1$ and $\varepsilon_i = 1$ for $i > 2$, $g = \Sigma_{i=1}^n \varepsilon_i b_i' \otimes b_i$. Then $gv = vg$ implies $\mu = 0$.

If (5) is true, assume for convenience $E \subset F$ is the kernel of $f' = \Sigma_1^{n+1} f_i'$, where $\delta(\{f_i\}_{i \leq n+1}) = 1$. Write $P = 1_F - (n+1)^{-1} f' \otimes f$. If $w \in L(E)$ commutes with each element of G, then wP commutes with each permutation of the f_i's, so $wP = \lambda 1_F + \mu P$ for some scalars λ, μ. Thus $w = wP|_E = (\lambda + \mu) 1_E$. □

§3. Applications and examples

EXAMPLE 3.1. Let E be R^n with the norm $\| x \| = \max_{i \neq k} | x_i + x_k |$, $n \geqq 13$. E satisfies (1) of Theorem 2.1.

PROOF. Let $B = \{b_i, b_i'\}_{i \leq n}$ be the unit vector basis for R^n, and Q the set of vectors $\pm(b_i' + b_k')$, $i \neq k$. Clearly Q consists of all the extreme points in the unit ball of E', therefore each isometry g of E must satisfy $g'(Q) = Q$. Of the five possibilities listed in Theorem 2.1, this is true only in case (1). □

EXAMPLE 3.2. Let E be R^n under the norm $\| x \| = \max\{| \Sigma_{j=1}^n x_j |, \max_{i \neq k} | x_i - x_k |\}$. For $n \geqq 13$, E satisfies (2) of Theorem 2.1.

PROOF. It is clear that the reflection about the sum of the unit vectors is an isometry, so this follows from Corollary 2.11. □

EXAMPLE 3.3. Let F be the subspace of R^n consisting of vectors with sum zero, normed by $\|x\| = \max_i |x_i|$. For $n \geqq 13$, F satisfies (5) of Theorem 2.1.

This follows easily from the next result.

THEOREM 3.4. *Let* $\dim(E) = n \geqq 14$, *and* $B = \{b_i, b'_i\}_{i \leqq n}$ *a basis satisfying* $\delta(B) = 1$. *Then either* $[b]^\perp$ *is isometric to a Hilbert space, or the group of isometries of* $[b]^\perp$ *is the group of operators* $\varepsilon \sum_{i=1}^n b'_i \otimes b_{\Pi(i)} \,|\, [b]^\perp$ $(\Pi \in S_n, |\varepsilon| = 1)$.

PROOF. If the group of isometries G of E is infinite this follows from Theorem 1.3, so we may assume G is finite, hence $[b]^\perp$ is not isometric to a Hilbert space. Since $s([b]^\perp) = 1$ and $\delta([b]^\perp) = 1$, the group of isometries G_0 of $[b]^\perp$ must be finite by Corollary 1.4. For $\Pi \in S_n$ write $g_\Pi(b_i) = b_{\Pi(i)}$, $P = n^{-1} b' \otimes b$ and let M be E under the norm

$$|x| = |\langle x, b' \rangle| + \|(1_E - P)x\|.$$

Clearly $|\ | = \|\ \|$ on $[b]^\perp$, also, if $g \in G_0$ and $|\varepsilon| = 1$ then $g(1_E - P) + \varepsilon P$ is an isometry of M. Taking $g = g_\Pi \,|\, [b]^\perp$ and $\varepsilon = 1$ shows $\delta(\{b_i\}_{i \leqq n} \subset M) = 1$, and taking $g = 1_{[b]^\perp}$ and $\varepsilon = -1$ shows that a scalar multiple of b is a reflection point of M. The group of isometries of M must be finite by Theorem 1.3, and by Corollary 2.11 M satisfies (2) of Theorem 2.1. The result follows immediately. □

Let $n = 2^m$, E be an n-dimensional normed space and $B = \{b_i, b'_i\}_{i \leqq n}$ a basis for E. The Haar basis built from B, denoted by $h(B)$, is the normalization of $b_1 - b_2$, $b_3 - b_4, \cdots, b_{n-1} - b_n$, $b_1 + b_2 - b_3 - b_4, \cdots, b_{n-3} + b_{n-2} - b_{n-1} - b_n, \cdots$, $\sum_{i=1}^n b_i (= b)$.

COROLLARY 3.5. *Let* $h(B)$ *be the Haar basis built from* B, *and assume* $\delta(B) = \chi(h(B)) = 1$. *Then* $[b]^\perp = l_2^{n-1}$.

PROOF. If $[b]^\perp$ is not isometric to l_n^{n-1} then by Theorem 3.4 every isometry of $[b]^\perp$ is the restriction of a g_Π—but no g_Π fixes $b_1 - b_2$ and changes the sign of $(b_1 + b_2 - b_3 - b_4)$, contradicting $\chi(h(B)) = 1$. □

In the following example E has a finite group of isometries yet contains l_2^{n-1} isometrically.

EXAMPLE 3.6. Let $B = \{b_i\}_{i \leqq n}$ be the unit vector basis of E, where E is R^n $(n \geqq 13)$ normed by:

$$\| x \| = \max \left\{ \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}, [n/(n-1)] \max_i | x_i | \right\}.$$

The group of isometries of E is finite.

PROOF. It is easy to check that

$$[n/(n-1)]^{1/2} = \inf \left\{ \left(\sum_{i \leqq n} x_i^2 \right)^{1/2} ; \max_i | x_i | = 1 \quad \text{and} \quad \sum_{i=1}^{n} x_i = 0 \right\},$$

so that $\| \ \|$ is Euclidean on $[b]^{\perp}$. E is clearly not isometric to l_2^n, and since $t(B) = 1$ it follows from Theorem 1.3 that the group of isometrics of E is finite. □

EXAMPLE 3.7. Let $n \geqq 6$ be even and Δ be the subgroup $\{\delta \in \{-1, 1\}^n ; \Pi_{i=1}^n \delta_i = 1\}$. Let $B = \{b_i, b_i'\}_{i \leqq n}$ be the unit vector basis of R^n, and let E be R^n under the norm $\| x \| = \max_{\delta \in \Delta} | \langle x, \delta \rangle |$. Then E satisfies (4) of Theorem 2.1.

PROOF. Observe first that $\| \ \|$ is a norm on R^n and that Δ is the set of extreme points of the unit ball of E'. Hence if $g \in G$, $g'(\Delta) = \Delta$. The first claim is that $g(\{b_1, \cdots, b_n\}) = \pm \{b_1, \cdots, b_n\}$ for any $g \in G$.

Let $x = g(b_i)$. Since $x \neq 0$ we may assume $x_1 > 0$. If $x_1 + x_k = 0$ for all $k \geqq 2$, then since $g'(\Delta) \subset \Delta$, $1 = | \langle b_1, g'(1, 1, \cdots, 1) \rangle | = (n-2) | x_1 |$ and $1 = | \langle b_1, g'(1, -1, -1, 1, \cdots, 1) \rangle | = (n-6) | x_1 |$, an impossibility. Thus $x_1 + x_k \neq 0$ for some $k \neq 1$, we may assume that $k = 2$. Let $\delta_3, \cdots, \delta_n$ be signs with $\Pi_3^n \delta_i = 1$. Then

$$1 = | \langle b_1, g'(1, 1, \delta_3, \cdots, \delta_n) \rangle | = \left| x_1 + x_2 + \sum_{3}^{n} \delta_i x_i \right|, \quad \text{and}$$

$$1 = | \langle b_1, g'(-1, -1, \delta_3, \cdots, \delta_n) \rangle | = \left| -(x_1 + x_2) + \sum_{3}^{n} \delta_i x_i \right|.$$

Since $x_1 + x_2 \neq 0$, $| x_1 + x_2 | = 1$ and $\Sigma_3^n \delta_i x_i = 0$, and letting δ_i vary on all possibilities, we have $x_i = 0$ if $i \geqq 3$. Also $1 = | \langle b_1, g'(1, -1, -1, 1, \cdots, 1) \rangle | = | x_1 - x_2 |$, which implies $x_1 = 1$, $x_2 = 0$.

Hence if $g \in G$, $g(b_i) = \varepsilon_i b_{\Pi(i)}$ for some $\Pi \in S_n$ and $| \varepsilon_i | = 1$. But $\Sigma_1^n \varepsilon_i b'_{\Pi^{-1}(i)} = g'(1, 1, \cdots, 1) \in \Delta$, so $\Pi \varepsilon_i = 1$. □

THEOREM 3.8. *Let $n = \dim(E) \geqq 13$ and $B = \{b_i, b_i'\}_{i \leqq n}$ be a basis for E.*

(i) *If B satisfies* (1) *or* (2) *of Theorem* 2.1, *then any diagonally symmetric basis for E must have the form $\{\lambda b_i + \mu b\}_{i \leqq n}$ where λ and μ are scalars, $b = \Sigma_1^n b_i$.*

(ii) *If B satisfies* (3) *or* (4) *of Theorem* 2.1, *then any diagonally symmetric basis* *for E must have the form $\{\lambda\varepsilon_i b_i + \mu \sum_{j=1}^n \varepsilon_j b_j\}_{i\leqq n}$ where λ and μ are scalars, $|\varepsilon_i| = 1$, $1 \leqq i \leqq n$.*

(iii) *If B satisfies* (3) *of Theorem* 2.1, *then any totally symmetric basis for E must have the form $\{\lambda\varepsilon_i b_i\}_{i\leqq n}$ where λ is a scalar, $|\varepsilon_i| = 1$, $1 \leqq i \leqq n$.*

PROOF. (i) If $\{e_i\}_{i\leqq n}$ is any diagonally symmetric basis for E, there are scalars α and β such that $\{\alpha e_i + \beta e\}_{i\leqq n}$ ($e = \sum_1^n e_i$) is orthonormal, so we may assume already that both bases $\{e_i\}_{i\leqq n}$ and $\{b_i\}_{i\leqq n}$ are orthonormal and diagonally symmetric. Since B satisfies (1) or (2), every reflection point is either $\pm n^{-1/2} b$ or $2^{-1/2}(b_i - b_j)$, $i \neq j$. Hence for any $i \neq j$, $2^{-1/2}(e_i - e_j)$ is either $\pm n^{-1/2} b$ or $2^{-1/2}(b_r - b_s)$. Assume, for example, $2^{-1/2}(e_1 - e_2) = n^{-1/2} b$, then $2^{-1/2}(e_1 - e_3) = 2^{-1/2}(b_r - b_s)$ for some $r \neq s$, but checking the scalar product of the two vectors yields that $(e_1 - e_2, e_1 - e_3) = 0$, which is a contradiction. Hence $\{e_i - e_j; i \neq j\} \subseteq \{b_i - b_j; i \neq j\}$.

Write $e_1 - e_2 = b_{i_1} - b_{i_2}$, $e_1 - e_3 = b_{j_1} - b_{j_2}$. Then $e_2 - e_3 = b_{j_1} - b_{i_1} + b_{i_2} - b_{j_2}$ hence either $j_1 = i_1$ and $j_2 \neq i_2$, or $j_1 \neq i_1$ and $j_2 = i_2$. Assume the first, then $e_1 - e_4$ must have, for similar reasons, the form $b_{i_1} - b_{j_3}$, and in general there is a permutation Π, $\Pi(1) = i_1$, such that $e_1 - e_k = b_{\Pi(1)} - b_{\Pi(k)}$, $k \geqq 2$. Then for each k, $0 = (e, e_1 - e_k) = (e, b_{\Pi(1)} - b_{\Pi(k)})$, so $e = \lambda b$, and from the equation

$$ne_j - e = \sum_{k=1}^n e_j - e_k = \sum_{k=1}^n b_{\Pi(j)} - b_{\Pi(k)} = nb_{\Pi(j)} - b$$

the required representation for e_j follows.

To prove (ii), note that in case (3), since $t(B) = 1$, B is orthogonal, and all b_i's have the same $\|\ \|_2$ norm, so we may assume the b_i's to be orthonormal. As in (i), passing to a sequence $\{\alpha e_i + \beta e\}$ if necessary, we shall assume that $\{e_i\}_{i\leqq n}$ is an orthonormal diagonally symmetric basis.

Since B satisfies (3), R consists of $\pm b_i$, $2^{-1/2}(b_i - b_j)$ and $\pm 2^{-1/2}(b_i + b_j)$, $1 \leqq i \neq j \leqq n$. Of course R contains the points $2^{-1/2}(e_i - e_j)$, $i \neq j$. We claim $2^{-1/2}(e_i - e_j)$ cannot have the form $\pm b_r$ for any $i \neq j$, $1 \leqq r \leq n$. Assume otherwise that for some $i \neq j$, $2^{-1/2}(e_i - e_j) = \varepsilon b_r$, $|\varepsilon| = 1$. Since for $k \neq j, i$, $(2^{-1/2}(e_i - e_j), 2^{-1/2}(e_i - e_k)) = 1/2$, it follows that $2^{-1/2}(e_i - e_k)$ cannot have the form $\pm b_s$, $2^{-1/2}(b_s - b_t)$, $\pm 2^{-1/2}(b_s + b_t)$ for any $s \neq t$, which is a contradiction.

Hence we have in case (3), and obviously also in case (4),

$$\{e_i - e_j; i \neq j\} \subseteq \{b_i - b_j, \delta b_i + \varepsilon b_j; i \neq j, |\varepsilon| = |\delta| = 1\}.$$

Write

$$e_1 - e_2 = \varepsilon_1 b_{i_1} + \delta_1 b_{j_1} \quad \text{where} \quad i_1 \neq j_1, |\varepsilon_1| = |\delta_1| = 1,$$

$$e_1 - e_3 = \varepsilon_2 b_{i_2} + \delta_2 b_{j_2} \quad \text{where} \quad i_2 \neq j_2, |\varepsilon_2| = |\delta_2| = 1.$$

Checking the inner product which is equal to 1, it follows that we may assume $i_1 = i_2$ and $\varepsilon_1 = \varepsilon_2$, and then necessarily $j_1 \neq j_2$. Writing $e_1 - e_4 = \varepsilon_3 b_{i_3} + \delta_3 b_{j_3}$ where $|\varepsilon_3| = |\delta_3| = 1$, $i_3 \neq j_3$, it follows by considering the inner product with the previous two vectors that $\varepsilon_3 = \varepsilon_1$ and $i_3 = i_1$, $j_3 \neq j_2$ and j_1.

Thus there exists $\Pi \in S_n$ and signs ε_i such that

$$e_1 - e_k = \varepsilon_1 b_{\Pi(1)} - \varepsilon_k b_{\Pi(k)} \qquad k = 2, 3, \cdots, n.$$

Since $(e, e_1 - e_k) = 0$ for all k, if we write $e = \Sigma_1^n \lambda_j b_{\Pi(j)}$, then $\lambda_k = \lambda_1 \varepsilon_1 \varepsilon_k$, and from $\| e \|_2 = n^{+1/2}$ it follows that $|\lambda_1| = 1$. Now

$$ne_1 - e = \sum_1^n e_1 - e_k = (n-1)\varepsilon_1 b_{\Pi(1)} - \sum_{k=2}^n \varepsilon_k b_{\Pi(k)}$$

so

$$ne_1 = (\lambda_1 + (n-1)\varepsilon_1) b_{\Pi(1)} - \sum_{k=2}^n \varepsilon_k (1 - \lambda_1 \varepsilon_1) b_{\Pi(k)}$$

and upon consideration of the norm $\| ne_1 \|_2 = n$, since $|\lambda_1| = 1$, it follows that $\lambda_1 = \varepsilon_1$, and so $e_1 = \varepsilon_1 b_{\Pi(1)}$ and $e_k = \varepsilon_k b_{\Pi(k)}$, $2 \leqq k \leqq n$. However, since the original symmetric basis had the form $\{\lambda e_k + \mu e\}$, the required representation follows.

(iii) If $t(\{e_i\}_{i \leqq n}) = 1$, then the e_i's have the same $\| \; \|_2$ norm and are orthogonal, so using the result proved in (ii), there is $\Pi \in S_n$ $(\varepsilon_i) \in \Delta_n$ so that $e_k / \| e_1 \|_2 = \varepsilon_k b_{\Pi(k)} / \| b_1 \|_2$, and the conclusion follows. □

THEOREM 3.9. *Let E be a normed space of even dimension $n \geqq 16$ which has no subspace of dimension $\geqq n-3$ isometric to a Hilbert space. If $B = \{b_i, b'_i\}_{i \leqq n}$ is a basis for E which satisfies* (4) *of Theorem* 2.1 *and if $\delta([b_1 - b_2]^\perp) = 1$, then $\chi(\{b_1 + b_2, b_3, \cdots, b_n\}) = 1$.*

PROOF. Write $F = [b_1 - b_2]^\perp$, assume $\delta(F) = 1$ and let G be the group of isometries of F. G is finite by Theorem 1.3. The restriction to F of the isometries $\Sigma_1^n \varepsilon_i b'_i \otimes b_{\Pi(i)}$, $|\varepsilon_i| = 1$ and $\Pi(i) = i$ for $i = 1, 2$, are distinct, so $|G| \geqq 2^{n-2}(n-2)!$, thus (1), (2) and (5) of Theorem 2.1 are impossible for G, and (4) is too since $\dim(F)$ is odd. Thus F satisfies (3) and $|G| = 2^{n-1}(n-1)!$. Let $[\,,]$ be the inner product generating the ellipsoid of least volume containing the unit ball of F.

Considering only isometries of F which are obtained from restricting to F

isometries of E, shows that the basis $b_1 + b_2, b_3, \cdots, b_n$ is $[\,,]$ orthogonal, and for purposes of the proof there is no loss of generality in assuming it to be $[\,,]$ orthonormal. Write $M = [b_3, \cdots, b_n] = \{x \in F; [x, b_1 + b_2] = 0\}$.

To obtain a contradiction assume the following to be true: If z is a reflection point of F, then either $z = \pm(b_1 + b_2)$ or $z \in M$. Then if z is any reflection point of F different from $\pm(b_1 + b_2)$, $z \in M$, so the operator $1 - 2z \otimes z$ maps M to M, is an isometry of F, therefore z is a reflection point of M. However, since $\{b_i\}_{i \leqq n}$ satisfied (4), $t(\{b_i\}_{i \geqq 3}) = 1$, hence M has exactly $2(n-2)^2$ reflection points (namely, $\pm b_i$, $\pm 2^{-1/2}(b_i - b_j)$, $\pm 2^{-1/2}(b_i + b_j)$, $3 \leqq i \neq j \leqq n$), therefore F has no more than $2(n-2)^2 + 2$ reflection points, contradicting the fact that F has $2(n-1)^2$ reflection points, because $t(F) = 1$.

Now let x be any reflection point of F satisfying

$$|[x, b_1 + b_2]| = \inf\{|[z, b_1 + b_2]|;\ z \text{ a reflection point of } F, [z, b_1 + b_2] \neq 0\}.$$

By the preceeding paragraph $0 < |x, b_1 + b_2]| < 1$. For $i \neq k \geqq 3$, $2^{-1/2}(b_i - b_k)$ is a reflection point of F, so we may assume by Lemma 2.5 that there is a constant a, a sign ε and integers m and s such that $[x, b_k] = a$ for $m \geqq k \geqq 3$, $[x, b_k] = a + \varepsilon 2^{1/2} \cos(\pi/s)$ for $m < k \leqq n$ and that either $s = 2$ and $m = n$; or, $s = 3$ and $m = 1, 2, n-4$ or $n-3$; or, $s = 4$ and $m = 1$ or $n-3$.

Either $m > 2$ or $n - m - 1 > 2$. For reasons of symmetry only the case $m > 2$ need be considered. By Lemma 2.2 there is an integer r such that $|[x, 2^{-1/2}(b_3 + b_4)]| = \cos(\pi/r)$. Thus

$$1 \geqq ma^2 = (m/2)\cos^2(\pi/r) \geqq (n-4)2^{-1}\cos(\pi/r),$$

and so $r = 2$ and $a = 0$.

Since $0 < |[x, b_1 + b_2]| < 1$ the only possible case is $s = 3$ and $m = n - 3$, so that $z = \sigma 2^{-1/2}(b_1 + b_2 + \varepsilon b_n)$ for some signs ε and σ. In any event there is an isometry of F mapping $b_1 + b_2$ to b_n, so that b_n, hence each b_k $(3 \leqq k \leqq n)$, is a reflection point of F. This shows that the basis $\{b_1 + b_2, b_3, \cdots, b_n\}$ consists of orthogonal reflection points and hence has unconditional constant one. □

Corollary 3.10. *For each $n \geqq 16$, there is an n-dimensional space E and a norm one complemented hyperplane $F \subset E$ such that $\delta(E) = 1$ and $\delta(F) > 1$.*

Proof. The space E of example 3.7 is polyhedral and hence satisfies the hypothesis of Theorem 3.9. But for $\{b_i\}_{i \leqq n}$ the unit basis, $\|\Sigma_{i=1}^{n} b_i\| = n$ and $\|\Sigma_{i=1}^{n-1} b_i - b_n\| = n - 2$, so $\chi(\{b_1 + b_2, b_3, \cdots, b_n\}) > 1$.

Thus $\delta([b_1 - b_2]^{\perp}) > 1$ and $F = [b_1 - b_2]^{\perp}$ is norm one complemented since $b_1 - b_2$ is a reflection point. □

REFERENCES

1. D. J. H. Garling and Y. Gordon, *Relations between some constants associated with finite-dimensional Banach spaces*, Israel J. Math. **9** (1971), 346–361.

2. Y. Gordon and D. R. Lewis, *Absolutely summing operators and local unconditional structures*, Acta Math. **133** (1974), 27–48.

3. Y. Gordon, *Unconditional Schauder decompositions of normed ideals of operators between some l_p spaces*, Pacific J. Math. **60** (1975), 71–82.

4. V. I. Gurarii, M. I. Kadec and V. I. Macaev, *On Banach–Mazur distance between certain Minkowsky spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **13** (1965), 719–722.

5. D. R. Lewis, *A Banach space characterization of the Hilbert* Schmidt class, to appear.

6. D. R. Lewis and P. Wojtaszczyk, *Symmetric bases in Minkowski spaces*, to appear.

7. S. Rolewicz, *Metric Linear Spaces*, Warszawa, 1972.

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